On primordial groups for the Green ring

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Abstract

Consider the Mackey functor assigning to each finite group G the Green ring of finitely generated kG-modules, where k is a field of characteristic $p > 0$. Thévenaz foresaw in 1988 that the class of primordial groups for this functor is the family of k-Dress groups. In this paper we prove that this is true for the subfunctor defined by the Green ring of finitely generated kG-modules of trivial source.

Keywords: Green ring, primordial group, trivial source module.

1 Introduction

For a field k, the Green ring of the category of finitely generated kG-modules, $a(kG)$, is by definition, spanned over $\mathbb{Z}$ by elements $[M]$, one for each isomorphism class of finitely generated kG-modules and with structures given by $[M] + [N] = [M \oplus N]$ and $[M][N] = [M \otimes_k N]$. The subring generated by the kG-modules of trivial source (defined in Section 2.1) is denoted by $a(kG, \text{triv})$.

Assigning to each finite group G either of the two above-mentioned rings defines a globally defined Mackey functor, as described in Bouc [3] and Webb [9]. These functors are denoted by $a(k\,\_)$ and $a(k\,\_, \text{triv})$. Based on Section 3 of Webb's paper [9], we suggest that the concept of primordial group can

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be defined for any globally defined Mackey functor. In terms of our two functors this concept is expressed in a familiar way: Let M be either $a(k\,\_)$ or $a(k\,\_,\mathrm{triv})$, then a group H is called primordial for M if $M(H)/T(H) \neq 0$ with

$$T(H) = \sum_{\substack{G \hookrightarrow H \\ G \not\cong H}} M(G)\!\uparrow_G^H .$$

This definition also works for the Mackey functor $G_0(k\,\_)$, which assigns to G the Grothendieck group of finitely generated kG-modules. We write $\mathrm{Prim}(M)$ for the class of primordial groups for M.

Primordial groups were first studied by Dress [6] in the context of Green functors for a finite group G. Thévenaz [8] proved that for such a functor N, the closure under conjugation and subgroups of the primordials for N is the minimal set $\mathscr{D}$ of subgroups of G satisfying $N(G) = \sum_{H \in \mathscr{D}} t_H^G N(H)$. Thévenaz also proved that if k is a field of characteristic $p > 0$, the primordial groups for $\mathbb{Q} \otimes a(kG)$ are the p-hypoelementary subgroups of G (see Section 2 for definitions). Also, he conjectured that the primordials for $G_0(kG)$ were the k-elementary subgroups of G, which was proved in 1989 by Raggi [7], and that an analogous of the Brauer-Berman-Witt Theorem (5.6.7, in Benson [1]) should hold for $a(kG)$, for which we give a close result in Theorem 3.2.

Recent work in the subject of induction can be found in the papers of Boltje [2] and Coşkun [4]. It is important to mention that some results about primordial groups can be generalized to the context of globally defined Mackey functors, as for example Lemma 3.1 that shows a general behavior of primordial groups of subfunctors.

2 Preliminaries

From now on, we assume that k is a field of characteristic $p > 0$, all modules are finitely generated, and all groups are finite.

Recall that for a group G and a prime r, $O^r(G)$ is defined as the smallest normal subgroup of G, such that $G/O^r(G)$ is a r-group, and $O_r(G)$ is the largest normal r-subgroup of G.

Definition 2.1. If q and r are primes, a group H is called *q-hyperelementary* if $O^q(H)$ is cyclic, and *r-hypoelementary* if $H/O_r(H)$ is cyclic.

Observe that H is q-hyperelementary if and only if $H = C \rtimes Q$ with Q a q-group and C a cyclic group of order prime to q, and it is r-hypoelementary

if and only if $H = D \rtimes C$ where D is a r-group and C is cyclic of order prime to r. It is easy to prove that the classes of q-hyperelementary and r-hypoelementary groups are closed under subgroups and quotients.

Notation 2.2. We write $\mathbb{Z}_m^*$ for the smallest non negative representatives of the multiplicative group of units modulo m (which we denote by $(\mathbb{Z}/m\mathbb{Z})^*$).

Definition 2.3. Suppose $H = C \rtimes Q$ is a q-hyperelementary group with $C =< x >$ or order m prime to p. The group H is called *k-elementary* if the action of every $y \in Q$ on x is given by $yxy^{-1} = x^a$ with $a \in I_m(k)$, where $I_m(k) \subseteq \mathbb{Z}_m^*$ is the set of smallest non negative representatives of the image of $\mathcal{G}al(k(\omega)/k)$ under the injective morphism

$$\begin{array}{rcl} \mathcal{G}al(k(\omega)/k) & \longrightarrow & (\mathbb{Z}/m\mathbb{Z})^* \\ \sigma & \longmapsto & \overline{a} \end{array}$$

if $\sigma(\omega) = \omega^a$ with $1 \leq a \leq m-1$, $(a, m) = 1$ and ω is a primitive m-th root of unity.

We write $\mathcal{E}_k$ for the class of k-elementary groups.

Note that in the latter definition we can replace $I_m(k)$ by $I_n(k)$ where n is any multiple of m.

Definition 2.4. For a prime q, a group H is called *q-Dress* if $O^q(H)$ is p-hypoelementary.

It is easy to see that $O^q(H)$ is p-hypoelementary if and only if $H/O_p(H)$ is q-hyperelementary. If in addition to this, $H/O_p(H)$ is k-elementary, then H is called *k-Dress* for q.

The class of q-Dress groups is closed under subgroups and quotients, and it is denoted by Dr_q. The class of groups which are k-Dress for some prime will be denoted by Dr_k. We write Dr_p^* for the class of k-Dress groups such that p divides the order of $H/O_p(H)$; that is, $H/O_p(H) = C \rtimes D$ is k-elementary with D being a nontrivial p-group.

2.1 Trivial source modules

The following facts on trivial source modules are well known and can be found in Benson [1] and Curtis [5].

Recall that for a kG-module it is equivalent to be $(G,\ H)$-projective and to be a direct summand of $L\uparrow_H^G$ for some kH-module L. If M is an indecomposable kG-module, there exists a subgroup D for which M is $(G,\ D)$-projective and $D \leqslant_G H$ for any subgroup H for which M is $(G,\ H)$-projective, such a group D is called a *vertex* of M. In this case, if L is an indecomposable kD-module such that M is a direct summand of $L \uparrow_D^G$, then L is called a *source* of M. The module M is said to have *trivial source* if the field k is a source of M, which is equivalent to say that M is a direct summand of a permutation module.

It can be proved that any two vertices of M are conjugate in G, and that any couple of sources of M are conjugate by an element in $N_G(D)$. Since we are assuming that k is a field of characteristic p, a vertex of M is a p-subgroup of G.

We prove the following property of trivial source modules, which will be used later.

Lemma 2.5. *The tensor induction of a trivial source module is a trivial source module.*

Proof. We denote the tensor induction from H to G by $\uparrow_H^{G^\otimes}$. Let B be a kH-module of trivial source, then it is a direct summand of a permutation module, say $\oplus_{a\in[H/K]}k = B \oplus A$. From the proof of Proposition 3.15.2 *iii*) in Benson [1], it is easy to see that the tensor induction of a permutation module is a permutation module. By the same proposition, on the right hand side we obtain $B\uparrow_H^{G^\otimes} \oplus A\uparrow_H^{G^\otimes} \oplus X$, where X is a sum of modules induced from proper subgroups of G. □

The following corollary to the Green Indecomposability Theorem, 19.22 in Curtis [5], will be used in the following sections, as well as the next lemma. Recall that k is a field of characteristic p.

Corollary 2.6 (19.23 in Curtis [5])**.** *Suppose that G is a p-group and H is an arbitrary subgroup. If L is an absolutely indecomposable kH module (i.e $k' \otimes L$ is indecomposable for all k' field extension of k), then $L \uparrow_H^G$ is an absolutely indecomposable kG-module.*

Lemma 2.7. *Let U be an indecomposable kH-module of trivial source such that one of its vertex contains $O_p(H)$, and let M be an indecomposable kG-module for $G \leqslant H$ such that U is a summand of $M\uparrow_G^H$. Suppose also that if $|H/O_p(H)|$ is divisible by p, we have M of trivial source. Then*

i) $O_p(H)$ *is contained in a vertex of* M *and it acts trivially on* M.

ii) *Any indecomposable summand* V *of* $M\uparrow_G^H$ *has trivial source,* $O_p(H)$ *is contained in a vertex of* V *and it acts trivially on* V.

Proof. Let D be a vertex of U that contains $O_p(H)$. If D_1 and S are a vertex and a source of M, respectively, then $O_p(H) \subseteq D_1 \subseteq G$ (because U is a direct summand of $S\uparrow_{D_1}^H$). If $|H/O_p(H)|$ is a p'-number then $O_p(H)$ is a vertex of U and $D_1 \subseteq O_p(H)$, so we have $O_p(H) = D_1$. This implies that S is a source of U and that M has trivial source. With this we obtain that V is of trivial source, which is the first part of *ii*).

Since M is a summand of $k\uparrow_{D_1}^G$, then $M\downarrow_{O_p(H)}^G$ is a direct summand of $k\uparrow_{D_1}^G\downarrow_{O_p(H)}^G$. By the Mackey formula, the latter is isomorphic to $\bigoplus_a k$, where a runs over $[G/D_1]$ so $M\downarrow_{O_p(H)}^G$ is isomorphic to a sum of k. With this we prove *i*), and by the same argument, we prove that $O_p(H)$ acts trivially on V.

Finally, if A is a vertex of V then $V\downarrow_{O_p(H)}^H \cong \bigoplus k$ is a summand of $k\uparrow_A^H\downarrow_{O_p(H)}^H$, which is isomorphic to $\bigoplus_b k\uparrow_{O_p(H)\cap A}^{O_p(H)}$. By Corollary 2.6, we have that $k\uparrow_{O_p(H)\cap A}^{O_p(H)}$ is indecomposable, then for some b we should have $k \cong k\uparrow_{O_p(H)\cap A}^{O_p(H)}$, So $O_p(H)$ is contained in A. □

3 Proof of the theorem

Lemma 3.1.

i) $Prim(\mathfrak{a}(k\_))$ *and* $Prim(\mathfrak{a}(k\_, triv))$ *are closed under subgroups and quotients.*

ii) $\mathcal{E}_k \subseteq Prim(\mathfrak{a}(k\_)) \subseteq Prim(\mathfrak{a}(k\_, triv)) \subseteq \bigcup_q Dr_q$.

Proof. i) The proof is the same for both functors, so M represents any of them. First, let G be a primordial group for M, and H a subgroup of G. By 2.5 we have tensor induction from $M(H)$ to $M(G)$, and clearly it sends the class of the field k to itself. Suppose that k can be written as a linear combination of modules induced from proper subgroups of H, then, by *iii*) and *iv*) of Proposition 3.15.2 in Benson [1], its image is a linear combination of modules induced from proper subgroups of G. This contradicts that G is primordial for M.

Now we take G/K, a quotient of G. Consider the inflation from $M(G/K)$ to $M(G)$. Again, the class of k is invariant under inflation, so if it could be written as a linear combination of modules induced from proper subgroups of G/K, then, these could be seen as modules induced form proper subgroups of G, which is a contradiction.

ii) In order to prove the inclusions

$$\mathcal{E}_k \subseteq \mathrm{Prim}(a(k\,\_)) \subseteq \mathrm{Prim}(a(k\,\_, \mathrm{triv}))$$

we recall that for every group H, we have the following morphisms

$$a(kH) \to G_0(kH) \quad \text{and} \quad a(kH, \text{triv}) \hookrightarrow a(kH),$$

the first one sends the class $[T]$ in $a(kH)$ to the class of T in $G_0(kH)$. These are morphisms of unitary algebras and commute with induction. To represent any of them we write $f_H : M(H) \to N(H)$. Now suppose that H is primordial for N. Given the properties of f_H, if k can be written as a linear combination of modules induced from proper subgroups of H in $M(H)$, then that can also be made in $N(H)$, which is a contradiction. So H is primordial for M. Recall that $\mathrm{Prim}(G_0(k\,\_)) = \mathcal{E}_k$, as mentioned in the Introduction.

To prove the inclusion $\mathrm{Prim}(a(k\,\_, \mathrm{triv})) \subseteq \bigcup_q Dr_q$, we will write M for $a(k\,\_, \mathrm{triv})$ and $\mathcal{D}$ for $\mathrm{Prim}(a(k\,\_, \mathrm{triv}))$. Let G be any group. We will use the following facts:

a) Using the Dress induction theorem for the Burnside ring, as stated in Yoshida's paper [10], we obtain a generalization of this theorem for the Mackey functor M. Let p-Hypo be the class of p-hypoelementary groups, we have:

$$M(G) = \sum_{\substack{K \leq G \\ K \in \mathcal{H}(p\text{-Hypo})}} M(K) \uparrow_K^G + \bigcap_{\substack{L \leq G \\ L \in p\text{-Hypo}}} \ker(res_L^G)$$

where res_L^G is the restriction from $M(G)$ to $M(L)$ and

$$\mathcal{H}(p\text{-Hypo}) = \{H \text{ group} \mid \exists q \text{ prime with } O^q(H) \in p\text{-Hypo}\}.$$

It is not hard to prove that $\mathcal{H}(p\text{-Hypo}) = \bigcup_q Dr_q$.

b) Observe that the Mackey functor $a(k\,\_)$ satisfies the Frobenius reciprocity formulas

$$(m \cdot (n \downarrow_K^G))\uparrow_K^G = (m\uparrow_K^G) \cdot n \quad \text{and} \quad ((n \downarrow_K^G) \cdot m)\uparrow_K^G = n \cdot (m\uparrow_K^G)$$

for all m in $a(kK)$ and n in $a(kG)$, with $K \leq G$.

c) Note also that $\mathcal{D}$ is the smallest class of groups closed under subgroups and quotients such that for every group G,

$$M(G) = \sum_{\substack{K \leq G \\ K \in \mathcal{D}}} M(K) \uparrow_K^G .$$

A proof of this is a slight modification of Thévenaz's in [8].

We consider the inclusion $a(kG, \text{triv}) \subseteq \mathbb{Q} \otimes a(kG)$ and we will write N for $\mathbb{Q} \otimes a(k\,\_)$. From the article of Thévenaz [8], we have that $\mathrm{Prim}(N)$ is the class p-Hypo and that

$$N(G) = \sum_{\substack{K \leq G \\ K \in p\text{-Hypo}}} N(K) \uparrow_K^G .$$

So we have

$$1_{M(G)} = 1_{N(G)} = \sum_{\substack{K \leq G \\ K \in p\text{-Hypo}}} n_K \uparrow_K^G$$

where $1_{M(G)}$ represents the unity of $M(G)$ and n_K is an element of $N(K)$. From the formula in a) we take an element m in $\bigcap_{\substack{L \leq G \\ L \in p\text{-Hypo}}} \ker(res_L^G)$, we have

$$\begin{aligned} m = 1_{M(G)} \cdot m &= \sum_{\substack{K \leq G \\ K \in p\text{-Hypo}}} (n_K \uparrow_K^G) \cdot m \\ &= \sum_{\substack{K \leq G \\ K \in p\text{-Hypo}}} (n_K \cdot (m \downarrow_K^G)) \uparrow_K^G \quad \text{by } b) \\ &= \sum_{\substack{K \leq G \\ K \in p\text{-Hypo}}} (n_K \cdot 0) \uparrow_K^G = 0. \end{aligned}$$

So we have

$$M(G) = \sum_{\substack{K \leq G \\ K \in \mathcal{H}(p\text{-Hypo})}} M(K) \uparrow_K^G .$$

Since $\mathcal{H}(p\text{-Hypo})$ is closed under subgroups and quotients, using c) we obtain $\mathrm{Prim}(a(k\,\_, \text{triv})) \subseteq \bigcup_q Dr_q$. ☐

Theorem 3.2. *Prim*$(a(k\,\_,$ *triv*$)) = Dr_k$.

The proof is given by Propositions 3.3 and 3.5.

With Proposition 3.5 we will conclude that every primordial group for $a(k\_)$ has to be k-Dress for some prime. On the other hand, the following proposition shows that every k-Dress group for a prime different from p is primordial for $a(k\_)$. As for the general case, the main difficulty arises from the fact that the techniques we use (namely Lemma 2.7) are ineffective for non trivial source modules.

Proposition 3.3.

i) $Dr_k \setminus Dr_p^* \subseteq Prim(a(k\_))$.

ii) $Dr_p^* \subseteq Prim(a(k\_, triv))$.

Proof. We prove $i)$ and $ii)$ simultaneously by contradiction. Let H be a k-Dress group, then we have two cases:

- $H/O_p(H)$ is not divisible by p; in this case we suppose H is not primordial for $a(k\_)$ to prove $i)$.
- $H/O_p(H)$ is divisible by p, so we assume H is not primordial for $a(k\_, \text{triv})$ to prove $ii)$.

In both cases, k can be written as a linear combination of modules induced from proper subgroups of H

$$k \oplus \left(\bigoplus_i M_i {\uparrow}_{L_i}^{H} \right) \cong \bigoplus_j N_j {\uparrow}_{T_j}^{H} \ .$$

In the first case, we can assume that M_i and N_j are trivial source modules. Notice that if an indecomposable module of trivial source is a direct summand of one $M_i \uparrow_{L_i}^{H}$ (or $N_j \uparrow_{T_j}^{H}$), then by Lemma 2.7, M_i (or N_j) and all its indecomposable summands are trivial source modules. Therefore, by the Krull-Schmidt Theorem, we can assume all of the M_i and N_j are of trivial source. Since in the second case we already assume that they are trivial source modules, the following arguments are valid for both cases. From 2.7, we have that $O_p(H)$ acts trivially on N_j and M_i and that they have a vertex containing $O_p(H)$. We take the quotients

$$k \oplus \left(\bigoplus_i M_i {\uparrow}_{L_i/O_p(H)}^{H/O_p(H)} \right) \cong \bigoplus_j N_j {\uparrow}_{T_j/O_p(H)}^{H/O_p(H)} \ .$$

This isomorphism is an equality in $a(k(H/O_p(H)), \text{triv})$, which is contained in $a(k(H/O_p(H)))$. Since $H/O_p(H)$ is k-elementary, Lemma 3.1 yields a contradiction. $\square$

Lemma 3.4. *If H is of smallest order being q-Dress and not k-Dress, then H is of the form*

$$H = < x > \rtimes < y > \quad \textit{where} \quad | < x > | = r, \quad | < y > | = q^n$$

with r and q different primes and $yxy^{-1} = x^a$ with $a \in \mathbb{Z}_r^* \setminus I_r(k)$.

Proof. Being a quotient of H, $H/O_p(H)$ is q-Dress but, since it is not k-elementary and $O_p(H/O_p(H)) = 1$, then $H/O_p(H)$ is not k-Dress. Therefore the minimality of H implies $O_p(H) = 1$. Hence, we have $H = C \rtimes Q$ with $C = < s >$ cyclic of order m and Q a q-group such that m not divisible by p and q. Now, as H is not k-elementary, there exists $y \in Q$ such that $ysy^{-1} = s^a$ with $a \in \mathbb{Z}_m^* \setminus I_m(k)$, so $H = C \rtimes C_{q^n}$, where C_{q^n} is the cyclic of order q^n generated by y.

Now we write $C = \prod_i C_{r_i}$, where C_{r_i} is the r_i-Sylow subgroup of C and the r_i are all the primes that divide m, so

$$C \rtimes C_{q^n} = \prod_i (C_{r_i} \rtimes C_{q^n}).$$

In addition, if ω is a primitive m-th root of unity, we have the commutative diagram

$$\begin{tikzcd} \mathcal{G}al(k(\omega)/k) \arrow[r, hook] \arrow[d, "\cong"'] & (\mathbb{Z}/m\mathbb{Z})^* \arrow[d, "\cong"] \\ \prod_i \mathcal{G}al(k(\omega^{m/r_i^{\alpha_i}})/k) \arrow[r, hook] & \prod_i (\mathbb{Z}/r_i^{\alpha_i}\mathbb{Z})^* \end{tikzcd}$$

where α_i is the largest positive integer such that $r_i^{\alpha_i}$ divides m. Thus, there exists i such that $C_{r_i} \rtimes C_{q^n}$ is not k-elementary. Rewriting $C_{r_i} = C_{r^\alpha}$ we have $H = C_{r^\alpha} \rtimes C_{q^n}$, where $C_{r^\alpha} = < x_o >$, $C_{q^n} = < y >$ and $yx_oy^{-1} = x_o^a$ with $a \in \mathbb{Z}_{r^\alpha}^* \setminus I_{r^\alpha}(k)$.

Finally, we take ζ a primitive r^α-th root of unity. We have

$$\mathcal{G}al(k(\zeta)/k) \cong \mathcal{G}al(k(\zeta)/k(\zeta^{r^{\alpha-1}})) \times \mathcal{G}al(k(\zeta^{r^{\alpha-1}})/k)$$

and $(\mathbb{Z}/r^\alpha\mathbb{Z})^* \cong A_{r^{\alpha-1}} \times A_{r-1}$ where these groups have order $r^{\alpha-1}$ and $r-1$ respectively. The morphism

$$\mathcal{G}al(k(\zeta)/k) \hookrightarrow (\mathbb{Z}/r^\alpha\mathbb{Z})^*$$

takes $\mathcal{G}al(k(\zeta)/k(\zeta^{r^{\alpha-1}}))$ into $A_{r^{\alpha-1}}$ and $\mathcal{G}al(k(\zeta^{r^{\alpha-1}})/k)$ into A_{r-1}, which is isomorphic to $(\mathbb{Z}/r\mathbb{Z})^*$, so we have the commutative diagram

$$\begin{tikzcd}
\mathcal{G}al(k(\zeta)/k) \arrow[r, hook] \arrow[d] & (\mathbb{Z}/r^\alpha\mathbb{Z})^* \arrow[d] \\
\mathcal{G}al(k(\zeta^{r^{\alpha-1}})/k) \arrow[r, hook] & (\mathbb{Z}/r\mathbb{Z})^*.
\end{tikzcd}$$

Now, we have $a^{q^n} \equiv 1 \bmod r^\alpha$, thus r does not divide the order of a modulo r^α, and we have $a \in \mathbb{Z}_r^*$. Since a is not in $I_{r^\alpha}(k)$, we have $a \in \mathbb{Z}_r^* \smallsetminus I_r(k)$. Taking $x = x_o^{r^{\alpha-1}}$ we have the result. $\square$

Proposition 3.5. *Prim*$(\mathcal{a}(k\,\_\,, \mathrm{triv})) \subseteq Dr_k$.

Proof. The proof is by contradiction. We suppose there is a group H of smallest order in $\mathrm{Prim}(\mathcal{a}(k\,\_\,, \mathrm{triv}))$ that is not k-Dress. Observe that the previous lemma is also valid if H satisfies a property that is preserved under subgroups and quotients and implies being q-Dress. Since this is the case for the property of being primordial for $\mathcal{a}(k\,\_\,, \mathrm{triv})$, the lemma gives us $H = C \rtimes Q$ with $C =< x >$ of order r and $Q =< y >$ of order q^n with r and q different primes and $yxy^{-1} = x^a$ with $a \in \mathbb{Z}_r^* \smallsetminus I_r(k)$.

We write 1_H to identify the field k as a kH-module. We shall prove that 1_H is a sum of modules induced from proper subgroups of H, contradicting the assumption of primordiality.

The image of 1_Q under the induction morphism $1_Q \uparrow_Q^H$ is isomorphic, as vector spaces, to $\bigoplus_{i=0}^{r-1} kx^i \otimes_Q 1_Q$. If ω is a primitive r-th root of unity, then the $k(\omega)H$-module $k(w) \otimes 1_Q \uparrow_Q^H$ as $k(w)$-vector space has the basis

$$\{x^i \otimes_Q 1_Q \mid i = 0, \dots, r-1\}.$$

We can define another basis $y_t := \sum_{i=0}^{r-1} \omega^{-ti}(x^i \otimes_Q 1_Q)$ for $t = 0, \dots, r-1$. To prove it is a basis, observe that the matrix

$$\begin{pmatrix}
1 & 1 & \dots & 1 & \dots & 1 \\
1 & \omega^{-1} & \dots & \omega^{-j} & \dots & \omega^{-(r-1)} \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
1 & \omega^{-(r-1)} & \dots & \omega^{-j(r-1)} & \dots & \omega^{-(r-1)^2}
\end{pmatrix}$$

has determinant equal to $\prod_{i \neq j}(\omega^{-i} - \omega^{-j})$, which is different from 0 since ω is a r-th primitive root of unity.

H acts on this basis in the following way

$$\begin{aligned} xy_t &= \sum_{i=0}^{r-1} \omega^{-ti}(x^{i+1} \otimes_Q 1_Q) \quad \text{if j=i+1,} \\ &= \sum_{j=0}^{r-1} \omega^{-t(j-1)}(x^j \otimes_Q 1_Q) = \omega^t y_t \\ yy_t &= \sum_{i=0}^{r-1} \omega^{-ti}(yx^i \otimes_Q 1_Q) = \sum_{i=0}^{r-1} \omega^{-ti}(x^{ai} \otimes_Q 1_Q) \\ &= \sum_{i=0}^{r-1} \omega^{-tbi}(x^i \otimes_Q 1_Q) = y_{t'} \end{aligned}$$

where $b \in \mathbb{Z}_r^*$ is such that $\overline{b}\overline{a} = 1$ in $(\mathbb{Z}/r\mathbb{Z})^*$, and $0 \leq t' \leq r-1$ satisfies $t' \equiv tb \bmod r$. From these relations we see that y_0 is fixed under the action of H, so $k(\omega)y_0$ is $k(\omega)H$-isomorphic to $k(\omega)$ and we have

$$k(\omega) \otimes 1_Q {\uparrow}_Q^H \cong k(\omega) \oplus \left(\sum_{t=1}^{r-1} k(\omega) y_t \right).$$

It is clear that $\mathcal{G} = \mathcal{G}al(k(\omega)/k)$ acts on $k(\omega) \otimes 1_Q {\uparrow}_Q^H$. Taking the fixed points of this action in the isomorphism above, gives us

$$1_Q {\uparrow}_Q^H \cong 1_H \oplus \left(\sum_{t=1}^{r-1} k(\omega) y_t \right)^{\mathcal{G}}.$$

Let σ be in $\mathcal{G}$, we write $b_\sigma \in I_r(k)$ for the integer through which σ is defined. We have

$$\sigma y_t = \sum_{i=0}^{r-1} \sigma(\omega^{-ti})(x^i \otimes 1) = \sum_{i=0}^{r-1} \omega^{-tb_\sigma i}(x^i \otimes 1) = y_s$$

where $0 \leq s \leq r-1$ and $s \equiv tb_\sigma \bmod r$. If $u = \sum_{t=1}^{r-1} \lambda_t y_t$ is in $(\sum_{t=1}^{r-1} k(\omega) y_t)^{\mathcal{G}}$, then for each $t \in \mathbb{Z}_r^*$ we must have $\sigma(\lambda_t) = \lambda_s$, where $s \in \mathbb{Z}_r^*$ and $s \equiv tb_\sigma \bmod r$. From this we define the vector spaces

$$M_l := \left\{ \sum_{\sigma \in \mathcal{G}} \sigma(\lambda_l) y_s \;\middle|\; s \equiv lb_\sigma \bmod r,\, \lambda_l \in k(\omega) \right\}$$

for each $l \in \mathbb{Z}_r^*$. Observe that $M_{l_1} = M_{l_2}$ if and only if $l_1 \equiv l_2 b_\sigma \mod r$ for some $\sigma \in \mathcal{G}$, this implies that

$$\left(\sum_{t=1}^{r-1} k(\omega)y_t\right)^{\mathcal{G}} = \bigoplus_{l \in \mathbb{Z}_r^*/I_r(k)} M_l.$$

We shall prove that the right hand side of this equality is a sum of modules induced from proper subgroups of H. We have $xM_l = M_l$ and $yM_l = M_{l'}$ with $l' \in \mathbb{Z}_r^*$ and $l' \equiv lb \mod r$. Since a does not belong to $I_r(k)$, neither does b, so M_l is never fixed under the action of y. Then M_l is not a kH-module. Taking the orbits of the action of y we have

$$\begin{aligned}\left(\sum_{t=1}^{r-1} k(\omega)y_t\right)^{\mathcal{G}} &= \bigoplus_{l \in \frac{\mathbb{Z}_r^*/I_r(k)}{\sim}} \left(\bigoplus_{z \in [H/A]} zM_l\right) \\ &= \bigoplus_{l \in \frac{\mathbb{Z}_r^*/I_r(k)}{\sim}} \left(M_l \uparrow_A^H\right)\end{aligned}$$

where $A = Stab_H(M_l)$, and $\sim$ represents the action of y.

It is clear that A is a proper subgroup of H. Finally, observe that every M_l is a trivial source kA-module. If $A = C_r \rtimes < y^d >$, then M_l is a direct summand of the induced module $\left(\sum_{\sigma \in \mathcal{G}} ky_s\right) \uparrow_{<y^d>}^A$, where $s \in \mathbb{Z}_r^*$ and $s \equiv lb_\sigma \mod r$. Since $ky_s \cong k$, then M_l has trivial source. $\square$

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